Fostering Consensus in Multidimensional Continuous Opinion Dynamics under Bounded Confidence

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1 Introduction

Consider a group of agents which is to find a common agreement about some issues which can be regarded and communicated as real numbers. Each agent has an opinion about each issue which he may change when he gets aware of the opinions of others. This process of changing opinions is a *process of continuous opinion dynamics*. Examples for discussing groups are parliaments, a commissions of experts or citizens in a participation process. The opinion issues in parliaments can be tax rates or items of the budget plan, in commissions of experts predictions about macroeconomic factors and for citizens the willingness to pay taxes or the commitment to a constitution.

In many processes of opinion dynamics it is desirable that the agents reach *consensus*, either for reaching a good approximation to the truth or for the reason, that reaching consensus is a good in itself (e.g. in the commitment to the constitution). Often, all relevant information about a societal issue has been collected and published but it is not reliable enough to bring a collective opinion or 'the truth' without opinion dynamics where agents judge, communicate and negotiate about the 'right' opinion. In the need of a collective decision it is the best for the group to achieve consensus because it does not need a decision by voting or other mechanisms with potential to conflict. In this study we make simple but reasonable assumption on humans in opinion dynamics. The models reproduce the formation of parties and interest groups and some other reasonable facts in real opinion dynamics. But there remain many reasonable free parameters of opinion dynamics, where we check a few with the aim to find structural conditions which might foster the achievement of consensus in the group.

We define the models based on two facts from social psychology. First, people adjust their opinions towards the opinions of others. This may be for normative or for informational reasons. So either because they feel conformational pressure and want to assimilate or because they appreciate the information of others to be relevant. Further on, people perceive themselves as members of a subgroup, according to the theory of self categorization. In our setting one feels as in a group with the people who have similar opinions. We put these descriptions of peoples behavior regarding opinion dynamics into rules for agents behavior: Agents find new opinions as averages of opinions of others and they will do this only with respect to agents which lie within their area of confidence.

Repeated averaging and bounded confidence lead to clustering dynamics. If the agents in our model have big enough areas of confidence they are able to find a consensus. If they are small they will fail and form several clusters. Are their structural properties of the opinion dynamics environment that have a positive effect on the chances of finding a consensus? Here, we will ask how structural properties of the opinion dynamics process as the communication regime, the number of opinion issues, their interdependence and the mode how agents form their area of confidence affect the chances for consensus?

With the question about conditions for consensus we grab an old research line of DeGroot [3] and Lehrer and Wagner [7] about the problem how to aggregate opinions to a rational consensus in science or society. They model aggregation by averaging with powers of reputation matrices. The work in [7] was in the flavor of the social choice problem. In recent times Hegselmann and Krause [4] grabbed on this with the idea of bounded confidence and formulated a model (now nonlinear) of opinion dynamics which can be seen as repeated meetings of agents with bounded confidence. Independently, Weisbuch, Deffuant and others [2, 11] formulated a similar bounded confidence model with random pairwise interaction, what we call gossip communication. They came with the background of social simulation, sociophysics and complexity science.

In section 2 we will outline and discuss the parameter space and define the two opinion dynamic processes. Section 3 shows the basic dynamics which are universal in these models: cluster formation in the time evolution and the bifurcation of cluster patterns in the evolution of the bound of confidence. We will set up on them in section 4 where we present and discuss the simulation results with a focus on the consensus transition. We show e.g. that raising the number of opinion issues fosters consensus if the issues are under budget constraints, but diminishes consensus if they are not. We conclude by giving a colloquial summary and pointing out further research directions.

2 Continuous Opinion Dynamics und Bounded Confidence

Here, we define the basic models of [4] and [11] such that they extend to more dimensional opinions and to different areas of confidence. We briefly discuss

real world interpretations.

The agents Often, analytical results are either possible for very low numbers of agents or in the limit for a large number of agents. Complexity arises with finite but huge numbers of agents. The fuzzy thing is that some macro level dynamics work, while at critical points changes appear very sensitive due to specific finite size effects. In the simulation studies we chose $n = 200$ because we regard this as applicable to a wide range of real groups of agents. We also checked $n = 50, 500$ to ensure that the results hold also in this range, which they do. This range of group sizes coincides with the social brain hypotheses [6] that humans can only hold about 150 relationships on average.

The opinion space and the initial profile The opinion space is the set of all possible opinions an agent may have. In continuous opinion dynamics about d issues this is $\mathbb{R}^d$. So, we call $x^i(t) \in \mathbb{R}^d$ the opinion of agent i and $x(t) \in (\mathbb{R}^d)^n$ the opinion profile at time $t \in \mathbb{N}$. The evolution of an opinion profile is the *process of continuous opinion dynamics*. Dynamics depend heavily on the initial opinion profile. If we model dynamics by repeated averaging, then dynamics take place in the convex region spanned by the initial opinion profile $x(0)$, we call this the *relevant opinion space*. For $d = 1$ this is always an interval. For higher d there are many shapes. In this study we will restrict us to $d = 1, 2, 3$ and two shapes of the initial relevant opinion space: the *cube* $\square^d := [0, 1]^d$ and the *simplex* $\triangle^d := \{y \in \mathbb{R}^{d+1}_{\geq 0} \mid \sum_{i=1}^{n} y_i = 1\}$ (see Fig. 1). Notice that $\triangle^d$ is a subset of $\square^{d+1}$. These two shapes stand for two different kinds of multidimensional problems. The cube represents opinions about d issues which can be changed independently. The simplex represents opinions about issues where the magnitude of one can only be changed by changing others in the other direction. The main example is a budget plan with a fixed amount of money to allocate. Further on, we restrict us to random initial opinions which are equally distributed in the relevant opinion space. (It is not trivial to produce an equal distribution on a simplex! Normalization to sum-one of a $d+1$-dimensional cube would be skewed. We produce it by taking a d-dimensional cube and throwing away all opinions with sum bigger than 1. Then we compute the missing least component for each opinion.)

The area of confidence The *area of confidence* is a region in the opinion space around an agent's opinion. He regards all opinions in this region as relevant and all others as irrelevant. This region moves when the agent changes his opinion. Formally, it is a compact and convex subset of the opinion space including the origin. The origin is mapped to the opinion of the agent. In a one dimensional opinion space the only relevant areas are intervals. In more dimensions several areas seem appropriate. We restrict this study to the unit balls of the 1- and the ∞-norm (see Fig. 1) centered on the opinion and scaled by a *bound of confidence* $\varepsilon > 0$. Thus, agents measure the distance of opinions $x^1, x^2 \in \mathbb{R}^d$ as $\left\|x^1 - x^2\right\|_1 = \sum_i \left|x^1_i - x^2_i\right|$ or as $\left\|x^1 - x^2\right\|_\infty = \max_i \left|x^1_i - x^2_i\right|$

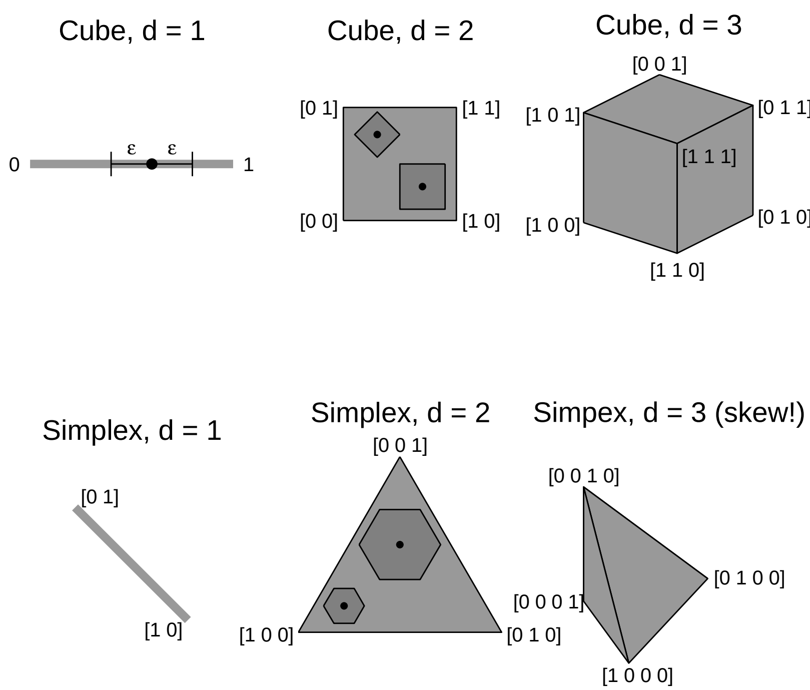


Figure 1: □ and △ opinion spaces with example areas of confidence for $p = 1, \infty$.

and judge their relevance by the threshold ε. We use these norms because they are close to how humans may judge differences in opinion. Agents using the 1-norm are willing to compensate between the opinion issues. If the other agent's opinion differs a lot in one issue this can be *compensated* by differing low in another issue. Agents using the ∞-norm are *noncompensators*. Their distance in each opinion issue should be below ε to accept another's opinion. For $d = 1$ the area is always an interval. For the cube and $d = 2, (3)$ the ∞-ball is a square (cube); for the 1-ball it is a diamond (octahedron). The intersection of the 2-dimensional simplex and the 3-dimensional area of confidence is a hexagon with edge length ε for the ∞-norm and with edge length $\varepsilon/2$ for the 1-norm. The intersection of the 3-dimensional simplex and the 4-dimensional area of confidence is an octahedron for the ∞-norm and a cuboctahedron for the 1-norm. Things get more fuzzy when going to more dimensions.

The communication regime The models of [4, 11] can both be extended naturally to the different opinion spaces and the areas of confidence outlined above. They differ in their communication regime. In the model of Hegselmann and Krause [4] each agent chooses his new opinion as the arithmetic mean of all opinions in his area of confidence. All agents do this at the same time. To do this, they need to know the opinions of all agents. We call it communication by *repeated meetings*. In the basic model of Deffuant, Weisbuch and others [11] two agents were chosen at random. They compromise in the middle if their opinions lie in the area of confidence of each other. We call this communication regime *gossip*.

Now we are ready for the mathematical definition of the two processes of continuous opinion dynamics.

Given an initial profile $x(0) \in \mathbb{R}^n$, a bound of confidence $\varepsilon > 0$ and a norm parameter $p \in \{1, \infty\}$ we define the *repeated meeting process* $(x(t))_{t\in\mathbb{N}}$ recursively through

$$x(t+1) = A(x(t), \varepsilon)x(t), \tag{1}$$

with $A(x, \varepsilon)$ being the *confidence matrix* defined

$$a_{ij}(x,\varepsilon) := \begin{cases} \frac{1}{\#I(i,x)} & \text{if } j \in I(i,x) \\ 0 & \text{otherwise,} \end{cases}$$

with $I(i, x) := \{j \mid \left\|x^i - x^j\right\|_p \leq \varepsilon\}$. ("#" stands for the number of elements.)

We define the *gossip process* as the random process $(x(t))_{t\in\mathbb{N}}$ that chooses in each time step $t \in \mathbb{N}$ two random agents i, j which perform the action

$$x^i(t+1) = \begin{cases} x^i(t) + \frac{1}{2}(x^j(t) - x^i(t)) & \text{if } \left\|x^i(t) - x^j(t)\right\|_p \leq \varepsilon \\ x^i(t) & \text{otherwise.} \end{cases}$$

The same for $x^j(t+1)$ with i and j interchanged.

Figure 2 demonstrates one time step in each process.

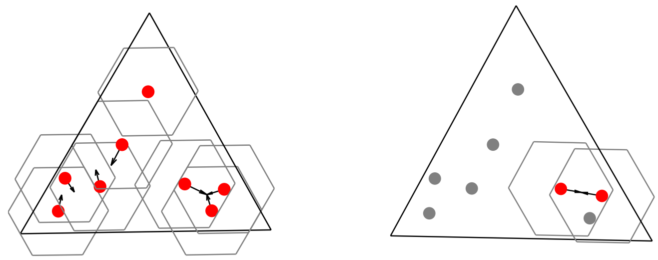


Figure 2: Examples of one step in meeting (left hand) and gossip (right hand) dynamics in the opinion space $\triangle^2$.

3 General Dynamics

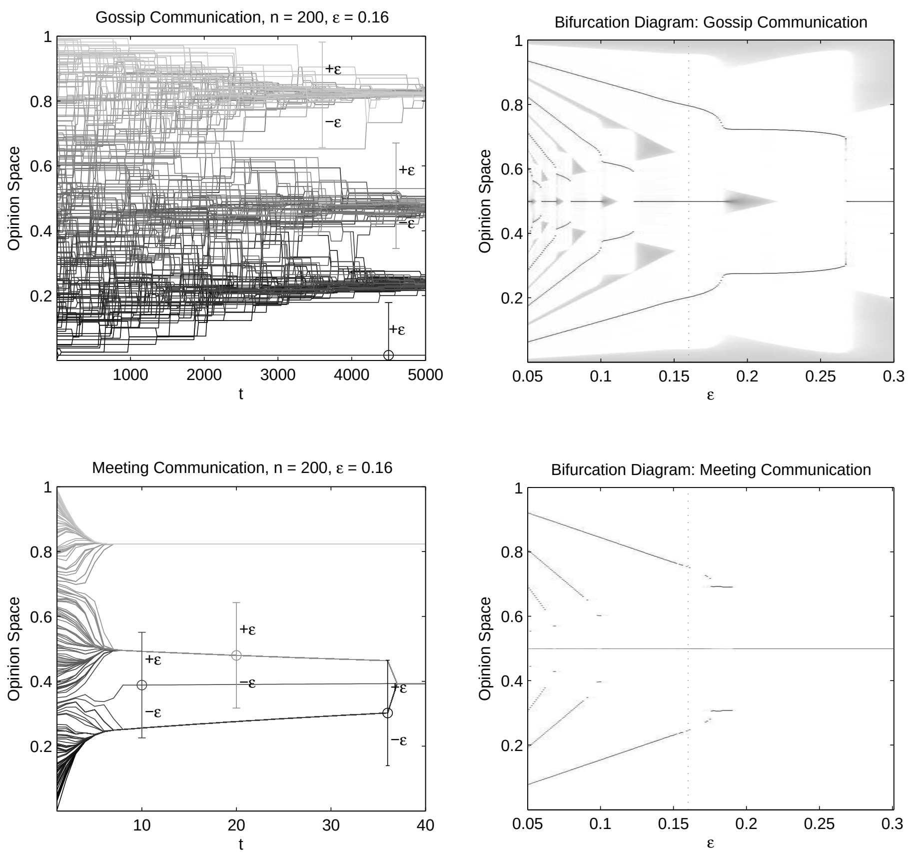
Clustering dynamics in the time evolution Every gossip and meeting process converges to a fixed configuration of opinion clusters [8, 10]. We call this fixed configuration the *stabilized profile*. A general dynamic is that opinion regions with high agent density attract agents from around. This attraction comes due to a higher probability to meet an agent in this region in gossip communication and due to the fact that the barycenter of opinions in an area of confidence is often close to a high density region.

If we consider an initial profile with uniformly distributed opinions on a certain relevant opinion space ($\square$ or $\triangle$) than the density distribution of opinions evolves over time as follows. (The following dynamic description can be traced in Fig. 3(a) for $\square^1$ and for $\triangle^2$ in Fig. 4.) Agents at the border of the relevant opinion space move closer to the center because opinions in their area of confidence are not equally distributed. Density in the center changes only due to random fluctuations in the initial conditions. So the relevant opinion space contracts but holds mainly the same shape but with a higher agent density at the border. If a more dimensional opinion space had some vertices (as $\square$ and $\triangle$ have) the density in the evolving high density regions is even higher at the vertices due to opinions coming from more sides.

These high density regions at the vertices of the relevant opinion space attract agents from the center and may get disconnected from the center and from the other vertices at some time, due to absorbtion of the connecting agents, and form a cluster. The dynamics goes on similar in the remaining cloud of connected opinions.

If some of these high density regions lie as close to each other that a small group of agents holds contact to both, then it may happen that they attract the agents in these high density regions and both join to form a bigger cluster. This may also happen to more clusters at the same time or with some delays (see Fig. 4 for an example). The fuzzy thing in more dimensions is that this contracting process happens on all face levels (e.g. faces and edges) of the shape of the opinion space on overlapping time scales. Further on, some clustering in the center may also occur due to slow deviations of uniformity. The time when

some high density regions have formed but have not completely disconnected from the rest is thus the critical time phase. In more than one dimension it is unpredictable which of the intermediate clusters joins with which others. Changes may happen due to very low fluctuations in the initial profile or the communication order.



(a) Example processes for gossip and meeting communication in the interval $[0, 1]$ demonstrating the time evolution to a stabilized profile. Notice one outlier for gossip and the meta-stable state in meetings.

(b) Reverse bifurcation diagrams of characteristic states of the stabilized profile in the ε-evolution. Diagrams derived by interactive Markov chains. Black is a high number of agents, gray a low number of agents.

Figure 3: Demonstration of general dynamical properties.

$\triangle^d$ has $d+1$ vertices and thus the same number of intermediate high density regions. The number of possible final cluster configurations that may evolve by disconnecting or joining of these high density regions is the same as the number of partitions of $\{1, \dots, d+1\}$ into pairwise disjoint subsets, which is the *Bell number* B_{d+1}. This shows the combinatorial explosion of different possible

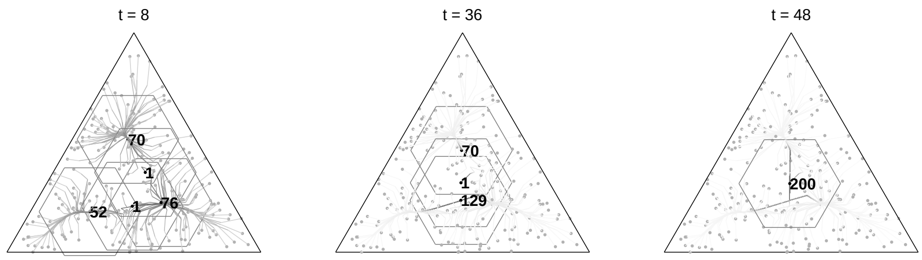


Figure 4: Example for meeting communication in $\triangle^2$ for interesting time steps. Notice the successive joining of intermediate clusters.

outcomes: $B_2 = 2, B_3 = 5, B_4 = 15, B_5 = 52, B_6 = 203, B_7 = 877, B_8 = 4140$.

Bifurcation dynamics in the evolution of the bound of confidence For each value of ε there is a certain *characteristic stabilized profile* under the assumption of a uniformly distributed initial profile. The number, the size and the location of opinion clusters in this stabilized profile are of interest. In Fig. 3(b) we see the reverse bifurcation diagrams for the attractive states of the meeting and the gossip process in $\square^1 = [0, 1]$ as relevant opinion space. These diagrams have been computed with interactive Markov chain that govern the evolution of the distribution of an idealized infinite population to a huge number of opinion classes in the opinion space (for details see [8, 9] and [1] for the inspiring differential equation approach). Such bifurcation diagrams should exist for moredimensional opinion spaces, too. A stabilized profile with 200 agents can be significantly blurred by low fluctuations in the initial profile and thus does not behave as the bifurcation diagram predicts. But as simulation shows, a bifurcation diagram with attractive states and certain discontinuous changes when manipulating ε seems to underlie opinion dynamics under bounded confidence.

It is easy to accept that $\varepsilon = 1$ leads to a central consensus, while $\varepsilon \to 0$ leads to full plurality where no opinion dynamic happens. The behavior in between can be understood as *bifurcations* of the consensual central cluster into other configurations of clusters. In the gossip and the meeting process the main effect when going down with ε is that the central cluster bifurcates at certain values of ε into two equally sized major clusters left and right which drift outwards when lowering ε further. The central cluster vanishes (nearly) completely to get reborn and grow again until it bifurcates again. We call the interval between two bifurcation points an *ε-phase* for a characteristic stabilized profile. The length of the ε-phases scales with ε, so for lower ε the phases get shorter. This fact is the basis of the $1/2\varepsilon$-rule (see [11]) which determines the number of major clusters under gossip communication.

Besides the common behavior the gossip and the meeting process differ. Under gossip communication there are minor clusters at the extremes, a nucleation of minor clusters between the central and the first off-central clusters and minor clusters between two major off-central clusters. These minor clusters occur as

a few outliers in agent based example processes, too. Meeting communication shows no minor clusters but the surprising phenomena of consensus striking back after bifurcation. Convergence in this phase takes very long (see [9]). The long convergence times to central consensus occurs also in front of each bifurcation of the central cluster. E.g. for $\varepsilon = 0.2$ we reach a meta-stable state of two off-central clusters and a small central cluster which attracts them very slowly to a consensus. The slow convergence due to meta-stable states close to bifurcation points occurs also in example processes.

In this study we focus on fostering consensus. So the most interesting point for us is the value of ε where the big central cluster bifurcates into two major clusters. This is the phase transition from polarization to consensus. We call this the *majority consensus transition*. Only 'majority' not total because of the extremal minor clusters in gossip communication. We call this point (in the style of [5]) the *majority consensus brink*.

4 Simulation Results

Simulation setup Our simulation setup deals with initial profiles of random and equally distributed opinions with 200 agents. We run processes for the 24 settings of the opinion spaces $\square, \triangle$ with dimensions $d = 1, 2, 3$, the areas of confidence for $p = 1, \infty$ and the communication regimes meeting and gossip. For each of these settings we took a big enough range of ε-values in steps of 0.01 so that we are sure that the majority consensus transition happens within this range. For each of this 24 settings and each value of the respective ε-range we run 250 simulation runs and collect the stabilized profiles for our final statistical analysis. We checked 50 and 500 agents with lower numbers of runs and verified that the results hold analog qualitatively and to a large extend quantitatively.

For each collection of stabilized profiles for a given point in the $\{\square, \triangle\}$-$\{d = 1, 2, 3\}$-$\{p = 1, \infty\}$-{meeting/gossip}-ε-parameter space, we have to measure the degree of consensus. In earlier studies the most used measure was the average number of clusters. This is inappropriate because of the minor clusters at the extremes under gossip communication. We use the *average size of the biggest cluster*. If it is 200 we are for sure above the majority consensus brink. If it is slightly below this can have two reasons according to what we know from section 3. First, some runs reach consensus, while some others polarize, or second, there is a big central cluster but also an amount of agents in minority clusters at the extremes. The second happens mostly for gossip communication.

Figure 5 shows the average size of the biggest cluster with respect to ε for all 24 parameter setting. We derive qualitative statements about *fostering consensus* from that. With fostering consensus we mean that the transition to a majority consensus appears for lower values of ε.

The impact of the communication regime (meetings vs. gossip) Communication in repeated meetings is fostering consensus in comparison to gossip. But surprisingly Fig. 6(a) gives strong evidence about the universal scale that

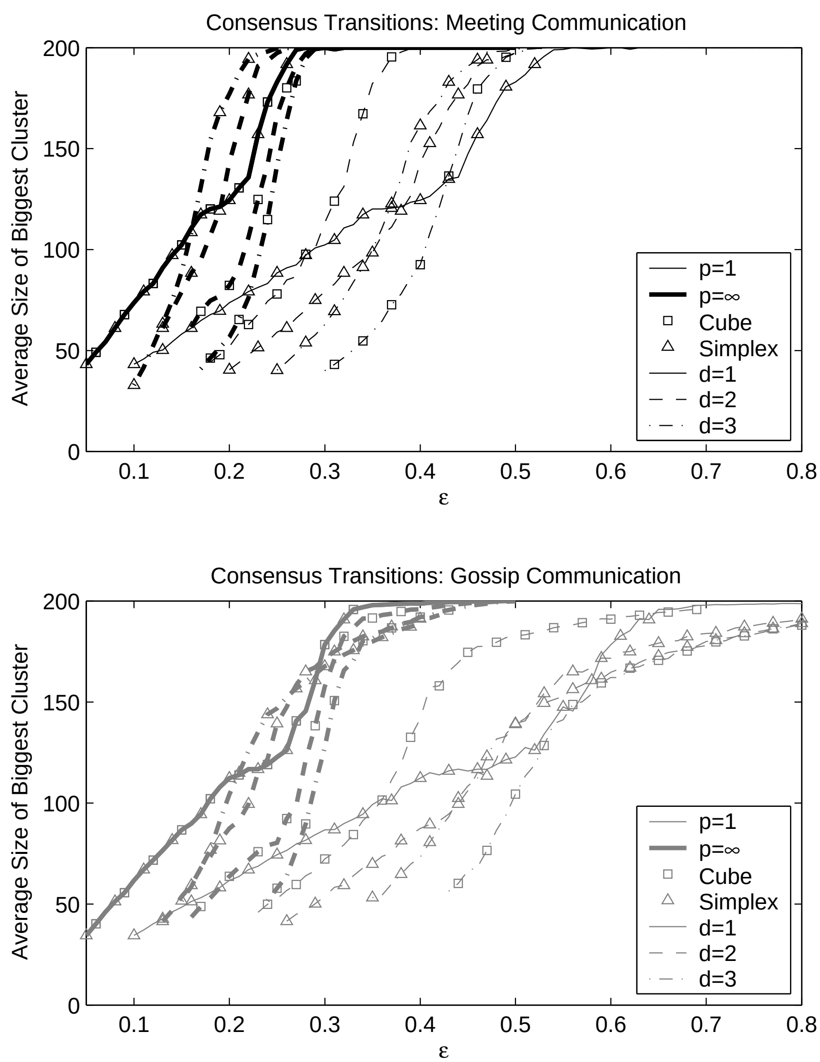
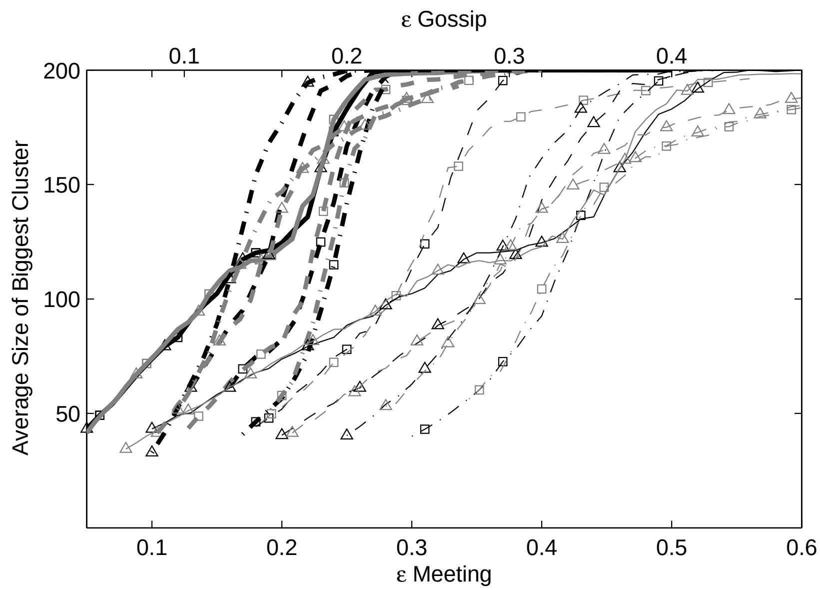
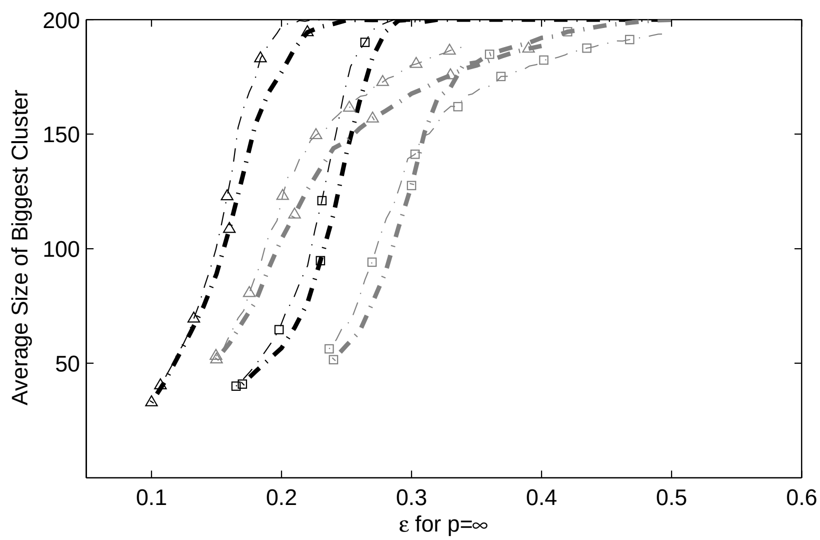


Figure 5: The average size of the biggest cluster for □, △ (marker), $d = 1, 2, 3$ (line style), $p = 1, \infty$ (line width) and communication regime (black, gray).



(a) The lines of Fig. 5 with ε-axis for lines of meeting communication scaled to 80%.



(b) The lines of Fig. 5 with *eps*-axis for lines of $p = 1$ scaled to volume equality with $p = \infty$.

Figure 6: Further simulation results for the average size of the biggest cluster.

a group of agents in meeting communication needs only 0.8ε to reach the same average size of the biggest cluster as the same group under gossip communication with ε. This holds for all our parameter settings. Only for very high sizes of the biggest cluster meeting communication gets even better, probably due to more minor clusters in gossip communication.

The impact of the number of opinion issues d What happens if we raise the number of issues? The answer is: It depends on the shape of the initial relevant opinion space. In a simplex, raising the number of issues fosters consensus. In a cube raising the number of issues dilutes consensus. Numerical values for fostering with meeting communication in a simplex and $p = \infty$: the biggest cluster contains at least 80% of the agents in 80% of the runs for $\varepsilon > 0.25$ with $d = 1$, $\varepsilon > 0.23$ with $d = 2$ and , $\varepsilon > 0.20$ with $d = 3$. One drawback is that under gossip communication we produce more and bigger extremal minor clusters in a simplex when raising d, one in each vertex. Thus, for fostering a complete consensus without outliers raising dimensionality under gossip dynamics is not good.

The impact of the shape of the relevant opinion space ($\triangle$ vs. $\square$) What fosters consensus better: an opinion space of three independent issues ($\square^3$) or four issues under fixed budget constraints ($\triangle^3$)? Colloquial: Is it good to add a budget dimension. The simplex is better for all p and all communication regimes. But this does not hold for $d = 2$, where the square is better under $p = 1$ but the simplex is better under $p = \infty$. Both shapes are trivially equal for $d = 1$. We conjecture that the simplex is getting better in higher dimensions. Another question of similar type is: Does it foster consensus to break a problem of three independent issues ($\square^3$) down to a problem of three issues under budget constraints ($\triangle^2$)? The answer is yes. It holds also for breaking down from $\square^2$ to $\triangle^1$ under $p = \infty$, but it is the other way round for $p = 1$.

The impact of compensating vs. noncompensating ($p = 1, \infty$) Imagine you appeal to your noncompensating ($p = \infty$) agents 'compensate: switch to $p = 1$'. This would imply that they should not tolerate distances of ε in each issue but only in the sum of all distances. Of course this will not foster consensus because their area of confidence is then only a smaller subset of their former. Perhaps you can appeal, that they should compensate in the way such that they should allow longer distances then ε in one issue in the magnitude as the other distances are short. This would lead to maximal distances of $d\varepsilon$ in one issue and perhaps the agent find this two much to tolerate. The 'mathematically correct' switching from noncompensating to compensating is to scale ε to that magnitude that the d-dimensional volumes of the areas of confidence would be equal. We did this for $d = 3$ in Fig. 6(b). The scale for $\square^3$ is $\sqrt[3]{6} \approx 1.82$ and for $\triangle^3$ it is $\sqrt[3]{64/5} \approx 2.34$. This 'normalization' leads to the result that switching to compensating fosters consensus a little bit. Probably this result holds only

in this configuration of the relevant opinion space and the area of confidence, there might be negative configurations.

5 Summary and Outlook

A colloquial summary: If we want to foster consensus and believe that agents adjust there opinions by building averages of other's opinions but have bounded confidence, then we should manipulate the opinion formation process in the following way (if possible):

- Install meetings (or publications) where everyone hears all opinions and do not rely only on gossip.
- Bring more issues in but put them under budget constraints.
- Release guidelines about compensation in the judgements of different issues.

Of course, our simple model neglects several properties of real opinion dynamics, e.g. rules about voting decisions, underlying social networks, heterogeneity of agent's confidence, long run ideologies or strategies and inflow of new information. All this are tasks for further analytical and experimental work. An unanswered question is also the reason for the universal 80% scale for meeting communication compared to gossip.

But we believe that under more realistic extensions there will be influence of the underlying bifurcation diagram and that critical consensus transitions will exist. Thus, it is worth to observe and design the structural properties of opinion dynamic processes, if one aims to foster consensus.

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